Axiomatic social choice theory

From Arrow's impossibility to Fishburn's maximal lotteries

ACM EC 2014 Tutorial

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DSS
Decision Sciences & Systems

Schedule

- 08.30 - 10.30: Tutorial part 1
- 10.30 - 11.00: Coffee break
- 11.00 - 12.30: Tutorial part 2

- In the interest of time, I cannot cover strategyproofness and preferences over lotteries.
  - If your are interested in these topics, you are welcome to attend the poster session and paper session 4b (both on Tuesday).

Motivation

- *What is "social choice theory"?*
  - How to aggregate possibly conflicting preferences into collective choices in a fair and satisfactory way?
  - Origins: mathematics, economics, and political science
  - Essential ingredients
    - Autonomous agents (e.g., human or software agents)
    - A set of alternatives (depending on the application, alternatives can be political candidates, resource allocations, coalition structures, etc.)
    - Preferences over alternatives
    - Aggregation functions
- The axiomatic method will play a crucial role in this tutorial.
  - Which formal properties should an aggregation function satisfy?
  - Which of these properties can be satisfied simultaneously?

Plurality

- Why are there different voting rules?
  - What's wrong with plurality (the most widespread voting rule) where alternatives that are ranked first by most voters win?
  - Consider a *preference profile* with 21 voters, who rank four alternatives as in the table below.

3	**5**	**7**	**6**
a	*a*	*b*	*c*
b	*c*	*d*	*b*
c	*b*	*c*	*d*
d	*d*	*a*	*a*

  - Alternative *a* is the unique plurality winner despite the fact that
    - a majority of voters think *a* is the worst alternative,
    - *a* loses against *b*, *c*, and *d* in pairwise majority comparisons, and
    - if the preferences of all voters are reversed, *a* still wins.

5 Common Voting Rules

- Plurality
  - Used in most democratic countries, ubiquitous
  - Alternatives that are ranked first by most voters
- Borda
  - Used in Slovenia, academic institutions, Eurovision song contest
  - The most preferred alternative of each voter gets *m-1* points, the second most-preferred *m-2* points, etc. Alternatives with highest accumulated score win.
- Plurality with runoff
  - Used to elect the President of France
  - The two alternatives that are ranked first by most voters face off in a majority runoff.

5 Common Voting Rules (ctd.)

- Instant-runoff
  - Used in Australia, Ireland, Malta, Academy awards
  - Alternatives that are ranked first by the lowest number of voters are deleted. Repeat until no more alternatives can be deleted. The remaining alternatives win.
  - In the UK 2011 alternative vote referendum, people chose plurality over instant-runoff.
- Sequential majority comparisons
  - Used by US congress to pass laws (aka *amendment procedure*) and in many committees
  - Alternatives that win a fixed sequence of pairwise comparisons (e.g., ((*a* vs. *b*) vs. *c*), etc.).

A Curious Preference Profile

33%	16%	3%	8%	18%	22%
a	*b*	*c*	*c*	*d*	*e*
b	*d*	*d*	*e*	*e*	*c*
c	*c*	*b*	*b*	*c*	*b*
d	*e*	*a*	*d*	*b*	*d*
e	*a*	*e*	*a*	*a*	*a*

Example due to Michel Balinski

- Plurality: ***a* wins**
- Borda: ***b* wins**
- Sequential majority comparisons (a,b,c,d,e): ***c* wins**
- Instant-runoff: ***d* wins**
- Plurality with runoff: ***e* wins**

Desirable Properties (Axioms)

- Anonymity
  - The voting rule treats voters equally.
  - Exchanging columns in the preference profile does not affect the outcome.
- Neutrality
  - The voting rule treats alternatives equally.
  - Renaming the alternatives does not affect the outcome.
- Monotonicity
  - A chosen alternative will still be chosen when it rises in individual preference rankings (while leaving everything else unchanged).
- Pareto-optimality
  - An alternative will not be chosen if there exists another alternative such that all voters prefer the latter to the former.
- Strategyproofness
  - No voter can obtain a more preferred alternative by misrepresenting his preferences.

	Anonymity	Neutrality	Monotonicity	Pareto	Strategy-proofness
Plurality	✓	✓	✓	✓	-
Borda	✓	✓	✓	✓	-
Plurality w/ runoff	✓	✓	-	✓	-
Instant-runoff	✓	✓	-	✓	-
SMC	✓	-	✓	-	-

1	**1**	**1**
a	*d*	*b*
b	*a*	*c*
c	*b*	*d*
d	*c*	*a*

SMC fails neutrality and Pareto-optimality

6	**5**	**4**	**2**
a	*c*	*b*	*b*
b	*a*	*c*	*a*
c	*b*	*a*	*c*

6	**5**	**4**	**2**
a	*c*	*b*	*a*
b	*a*	*c*	*b*
c	*b*	*a*	*c*

Runoff rules fail monotonicity

Outline

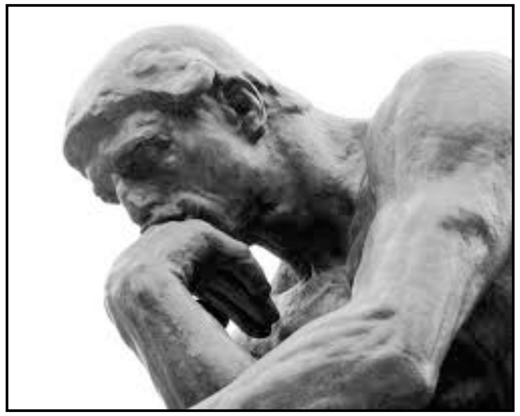
- Rational choice theory
- May's Theorem, Condorcet's Paradox, Arrow's Theorem
- Three escape routes:
  - replace consistency with a variable-electorate condition
    - scoring rules (e.g., plurality, Borda)
  - weaken consistency
    - top cycle, uncovered set, Banks set, tournament equilibrium set
  - randomization
    - maximal lotteries

Choice Theory

- A prerequisite for analyzing collective choice is to understand individual choice.
- Let U be a finite universe of alternatives.
- A *choice function* S maps a feasible set $A \subseteq U$ to a choice set $S(A) \subseteq A$.
  - We require that $S(A)=\varnothing$ only if $A=\varnothing$.
  - For simplicity, we will focus on resolute (i.e., single-valued) choice functions for now.
- Not every choice function complies with our intuitive understanding of rationality.
  - Certain patterns of choice from varying feasible sets may be deemed inconsistent, e.g., choosing a from $\{a,b,c\}$, but b from $\{a,b\}$.

X	***S(X)***
ab	*a*
bc	*b*
ac	*a*
abc	*a*

Preference and Maximality



- Rational decision-making process (note the order)
  - What is desirable?
  - What is feasible?
  - Choose the most desirable from among the feasible.
- Binary preference relation R on U
  - xRy is interpreted as *"x is at least as good as y"*
  - For simplicity, we assume that R is asymmetric and complete: for all $x \neq y$, either xRy or yRx.
- Best alternatives
  - For a binary relation R and a feasible set A, $Max(R,A) = \{x \in A : \text{there is no } y \text{ such that } yRx \text{ and not } xRy\}$

Rationalizable Choice

- S is rationalizable if there exists a binary relation R on U such that

$$S(A)=Max(R,A) \text{ for all } A.$$

  - A natural candidate for such a relation is the base relation R_S:
    $x \, R_S \, y \Leftrightarrow x \in S(\{x,y\})$
  - In fact, S can only be rationalized by its base relation R_S, which furthermore has to be transitive when S is resolute (as otherwise $Max(R,A)$ may be empty).
- The previously mentioned choice function S with $S(\{a,b,c\})=\{a\}$ and $S(\{a,b\})=\{b\}$ cannot be rationalized.

Consistency

- It would be a nice if irrationality (i.e., the non-existence of a rationalizing relation) could be pointed out by finding inconsistencies.
  - Consistency conditions directly relate choices from variable feasible sets with each other.
- A resolute choice function S satisfies consistency if for all A,B with $S(A) \subseteq B \subseteq A$ implies $S(B)=S(A)$.
  - If x is chosen in a feasible set, then it is also chosen in all subsets that contain x.
    - Example: Plurality does not satisfy consistency (when scores are computed for each feasible set).
    - $S(\{a,b,c\}) = \{a\}$ and $S(\{a,b\}) = \{b\}$

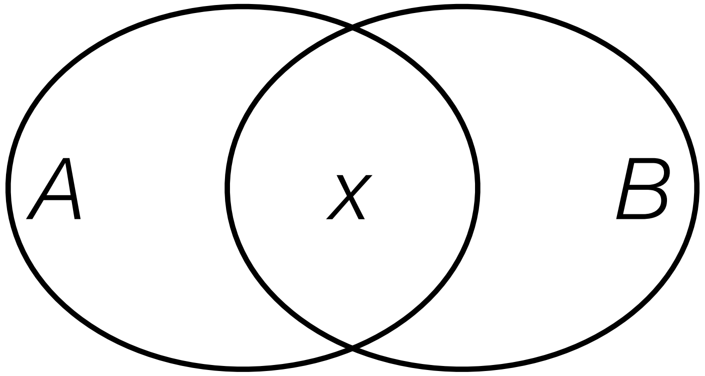
3	**2**	**2**
a	b	c
b	c	b
c	a	a

Rationalizability and Consistency

Amartya K. Sen

- Theorem (Sen, 1971): A resolute choice function is rationalizable iff it satisfies consistency.
- For resolute choice functions, consistency is equivalent to Samuelson's weak axiom of revealed preference (WARP) and the following condition due to Schwartz (1976):

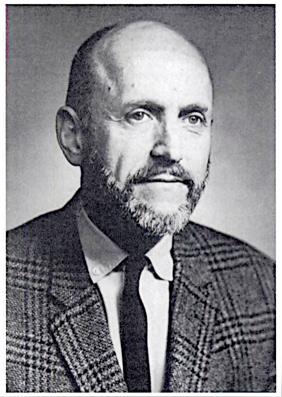
  For all A,B and $x \in A \cap B$, $x \in S(A \cup B) \Leftrightarrow x \in S(A) \cap S(B)$



From Choice to Social Choice

- N is a finite set of at least two voters.
  - For simplicity, we will assume $|N|$ is odd whenever possible.
- $R(U)$ is the set of all transitive, complete, and anti-symmetric relations over U.
- Every $R_N=(R_1, \ldots, R_{|N|}) \in R(U)^N$ will be called a *preference profile*.
- A *social choice function (SCF)* is a function f that assigns a choice function to each preference profile.
  - We will write $f(R_N,A)$ as a function of both R_N and A.
  - Rationalizability and consistency conditions carry over to SCFs.

May's Theorem



Kenneth May

- We first restrict attention to feasible sets of size two.
  - Let $n_{xy} = |\{i \in N: x\ R_i\ y\}|$ and define the *majority rule relation* as $(x\ R_M\ y) \Leftrightarrow n_{xy} > n_{yx}$.
  - The *majority rule SCF* is define as $f(R_N, \{x,y\}) = Max(R_M, \{x,y\})$.
- Theorem (May, 1952): Majority rule is the only resolute SCF on two alternatives that satisfies anonymity, neutrality, and monotonicity.
- Majority rule is very uncontroversial.
  - All voting rules mentioned earlier coincide with majority rule on two alternatives.
  - Majority rule is strategyproof.

The Condorcet Paradox


Marquis de Condorcet

- Theorem (Condorcet, 1785; May, 1952): No anonymous, neutral, and monotonic resolute SCF is rationalizable when $|U|\geq 3$.
  - Proof sketch: Let f be an SCF with the desired properties and consider the following preference profile.
  - May's theorem implies that $R_f = R_M$.
  - R_M is cyclic and therefore cannot rationalize f.

1	1	1
a	b	c
b	c	a
c	a	b

- Alternative x is a *Condorcet winner* in A if $x \; R_M \; y$ for all $y \in A$.
  - Condorcet winners may not exist, but whenever they do they are unique.
- Arrow's theorem can be obtained by significantly weakening anonymity, neutrality, and monotonicity.

From Condorcet to Arrow

- An SCF satisfies *independence of infeasible alternatives (IIA)* if the choice set only depends on preferences over alternatives within the feasible set.
- An SCF satisfies *Pareto-optimality* if an alternative will not be chosen if there exists another alternative such that all voters prefer the latter to the former.
- An SCF f is *dictatorial* if there exists a voter whose most preferred alternative is always uniquely chosen.
- These conditions can be formally defined such that
  - IIA is weaker than neutrality,
  - Pareto-optimality is weaker than monotonicity, and
  - non-dictatorship is weaker than anonymity.

Arrow's Impossibility

Kenneth J. Arrow

- Theorem (Arrow, 1951): There is no SCF that satisfies IIA, Pareto-optimality, non-dictatorship, and rationalizability when $|U| \geq 3$.
- Arrow's theorem is usually presented in an alternative formulation for social welfare functions, i.e., functions that aggregate individual preference relations into a collective preference relation.
  - IIA, Pareto-optimality, and non-dictatorship can be appropriately redefined for SWFs (by considering the base relation).
  - Theorem (Arrow, 1951): Every SWF that satisfies IIA and Pareto-optimality is dictatorial when $|U| \geq 3$.

What now?

- Rationalizability (or, equivalently, consistency) is incompatible with collective choice when $|U| \geq 3$.
  - Dropping IIA offers little relief (Banks, 1995).
  - Dropping Pareto-optimality offers little relief (Wilson, 1972).
  - Dropping non-dictatorship is unacceptable.
- A classic escape from Arrow's impossibility is to consider restricted domains of preferences in which majority rule is transitive (such as single-peaked preferences).
- In this tutorial, we will consider three other escape routes:
  - replace consistency with a variable-electorate condition
  - weaken consistency
  - randomization

Escape Route #1

Replace consistency
with a variable-electorate condition

Borda vs. Condorcet

- Jean-Charles Chevalier de Borda (1733 – 1799)
  - Mathematician, physicist, and sailor
  - Participated in the construction of the standard-meter (1/10,000,000 of the distance between the north pole and the equator)
- Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743 – 1794)
  - Philosopher and mathematician
  - Early advocate of equal rights and opponent of the death penalty

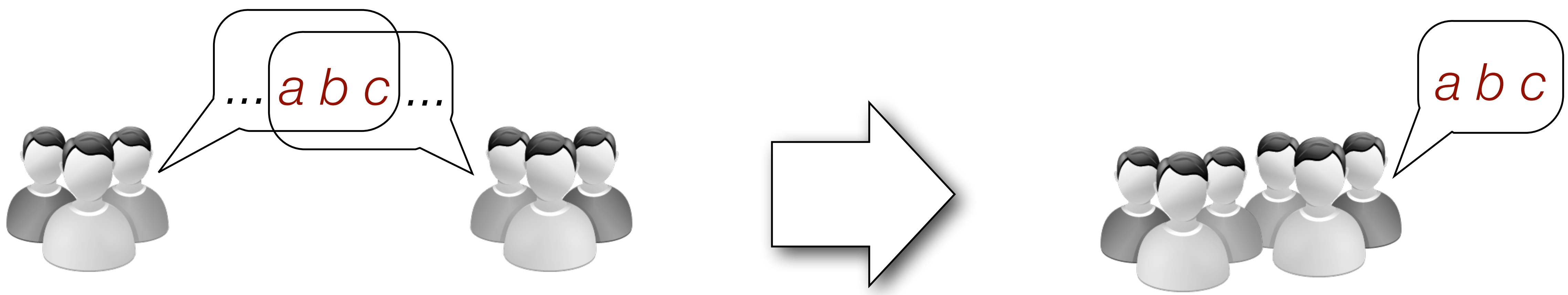
Scoring Rules and Condorcet Extensions

- Fix the feasible set A and let $|A|=m$.
  - A *score vector* is a vector $s=(s_1, \ldots, s_m)$ of real numbers.
  - If a voter ranks an alternative at the ith position, it gets s_i points.
- A *scoring rule* chooses those alternatives for which the accumulated score is maximal.
- Examples
  - Borda's rule: $s=(m-1, m-2, \ldots, 0)$
  - plurality rule: $s=(1, 0, \ldots, 0)$
- An SCF f is a *Condorcet extension* if $f(R_N,A)=\{x\}$ whenever x is a Condorcet winner in A according to R_N.
  - Example (Copeland's rule): Choose those alternatives that win most pairwise majority comparisons.

Scoring Rules and Condorcet Extensions

- When $|U|=2$, majority rule is the only monotonic resolute scoring rule and the only Condorcet extension.
- Proposition (Condorcet, 1785): Borda's rule is no Condorcet extension when $|U|\geq 3$.
- Theorem (Fishburn, 1973): No scoring rule is a Condorcet extension when $|U|\geq 3$.
- Theorem (Smith, 1973): A Condorcet winner is never the alternative with the lowest Borda score. Borda's rule is the only scoring rule for which this is the case.

Variable Electorates



- One of the most remarkable results in social choice theory characterizes scoring rules in terms of a variable set of voters ("electorates").
- *Reinforcement*
  - All alternatives that are chosen simultaneously by two disjoint electorates are precisely the alternatives chosen by the union of both electorates.

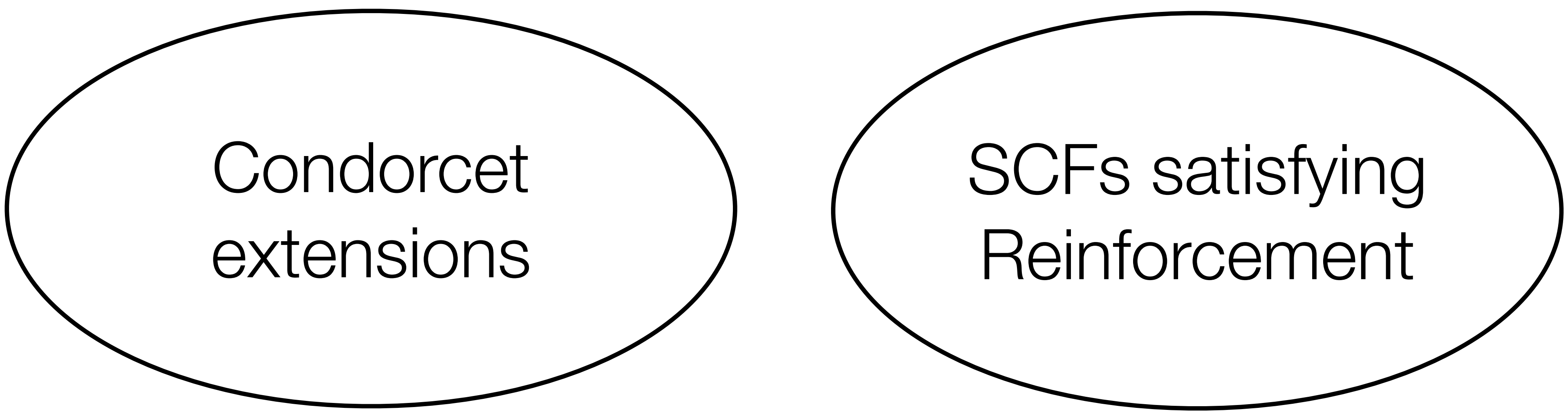
Characterization of Scoring Rules

H. Peyton Young

- Reinforcement is the equivalent of consistency for a variable electorate!
  - Consistency: $x \in f(R_N,A) \cap f(R_N,A') \Leftrightarrow x \in f(R_N,A \cup A')$ [$x \in A \cap A'$]
  - Reinforcement: $x \in f(R_N,A) \cap f(R_{N'},A) \Leftrightarrow x \in f(R_N \cup R_{N'},A)$ [$f(R,A) \cap f(R',A) \neq \emptyset$]
- Theorem (Smith, 1973; Young, 1975): A neutral and anonymous SCF is a scoring rule iff it satisfies continuity and reinforcement.
  - Loosely speaking, an SCF satisfies continuity if negligible fractions of voters have no influence on the choice set.
  - Continuity is a technical axiom that can be dropped when fixing an upper bound on the number of voters.
  - Reinforcement is the defining property of scoring rules.

The Dilemma of Social Choice

- Theorem (Young and Levenglick, 1978): No Condorcet extension satisfies reinforcement when $|U| \geq 3$.
  - Two centuries after Borda and Condorcet, the rationales between both ideas were shown to be incompatible.



- When aggregating preference relations to *sets of preference relations*, the intersection of these two sets contains exactly one neutral function: Kemeny's rule!

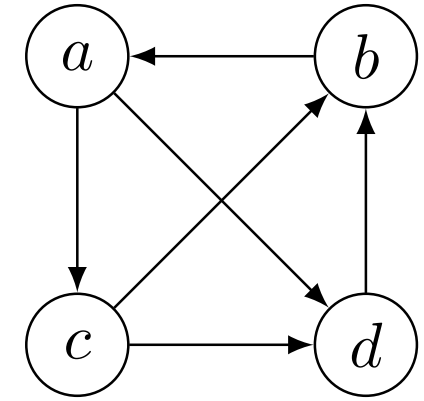
Escape Route #2

Weaken consistency

Majoritarian SCFs

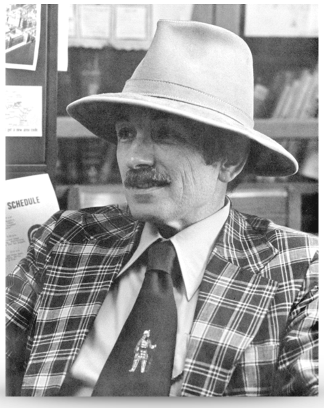
- An SCF is *binary* if the choice set only depends on the pairwise choices within the feasible set.
  - Binariness is stronger than IIA.
- A *majoritarian* SCF is an SCF that satisfies anonymity, neutrality, monotonicity, and binariness.
  - The choice set only depends on the base relation, which is furthermore fixed to be majority rule (May's theorem).
- Majoritarianness strengthens all conditions from Arrow's theorem except rationalizability/consistency.
- Weakening consistency allows us to uniquely characterize appealing SCFs.

Tournaments



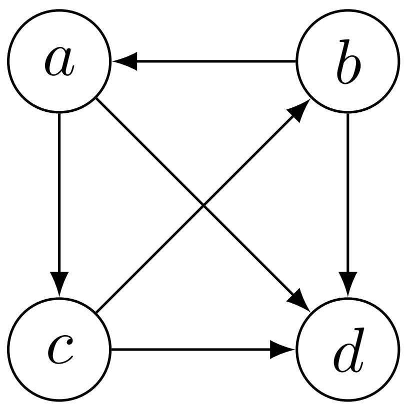
- For a given preference profile, majority rule R_M and a feasible set A define a tournament (A,R_M), an oriented complete graph.
  - We say that b dominates a if $b\ R_M\ a$.
  - Every tournament is induced by some preference profile (McGarvey's, 1953).
- We will write majoritarian SCFs as functions of tournaments (A,R_M) rather than functions of (R_N,A).
  - SCF f is said to be *finer* than SCF g if $f \subseteq g$.
  - Dominion $D(x)=\{y \in B \mid x\ R_M\ y\}$
  - Dominators $\overline{D}(x)=\{y \in B \mid y\ R_M\ x\}$

The Top Cycle

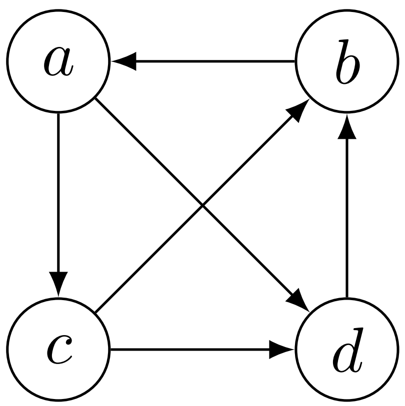

John I. Good

- Consistency can be weakened to expansion:
  $B \subseteq A$ and $S(A) \cap B \neq \emptyset$ implies $S(B) \subseteq S(A)$.
- Theorem (Bordes, 1976): There is a unique finest majoritarian SCF satisfying expansion: the top cycle.
- A *dominant set* is a nonempty set of alternatives $B \subseteq A$ such that for all $x \in B$ and $y \in A \setminus B$, $x \; R_M \; y$.
  - The set of dominant sets is totally ordered by set inclusion (Good, 1971).
  - Hence, every tournament contains a unique minimal dominant set called the top cycle (*TC*).
  - Also known as *GETCHA* (Schwartz, 1986) or *Smith set* (Smith, 1973)
  - *TC* is a Condorcet extension.

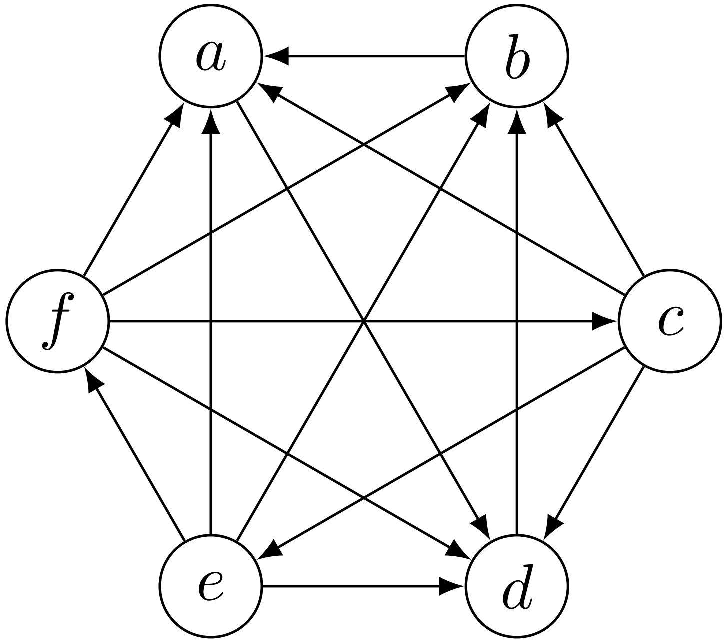
Examples



$TC(A,R_M)=\{a,b,c\}$



$TC(A,R_M)=\{a,b,c,d\}$



$TC(A,R_M)=\{c,e,f\}$

TC Linear-Time Algorithm

- Algorithm for computing TC_x, the minimal dominant set containing a given alternative x
  - Initialize working set B with $\{x\}$ and then iteratively add all alternatives that dominate an alternative in B until no more such alternatives can be found.
  - Computing TC_x for every alternative x and then choosing the smallest set yields an $O(m^3)$ algorithm where $m=|A|$.
    - A *linear-time* algorithm is $O(m^2)$ because the input is quadratic in m.
- Alternatives with maximal degree are always contained in TC (and linear-time computable).
- Hence, we only need to compute TC_x for some x with maximal degree.

procedure $TC(A, P_M)$
  $B \leftarrow C \leftarrow CO(A, P_M)$
  loop
    $C \leftarrow \bigcup_{a \in C} \overline{D}_{A \setminus B}(a)$
    if $C = \emptyset$ **then return** B **end if**
    $B \leftarrow B \cup C$
  end loop

Transitive Closure

- The essence of Condorcet's paradox and Arrow's impossibility is that the base relation fails to be transitive.
  - Why not just take the transitive (reflexive) closure R_M^*?
- Theorem (Deb, 1977): $TC(A,R_M) = Max(R_M^*,A)$.
- Consequences
  - R_M^* rationalizes the top cycle.
  - *TC* itself is a cycle. It is the source component in the DAG (directed acyclic graph) of strongly connected components.
  - Alternative linear-time algorithms using Kosaraju's or Tarjan's algorithm for finding strongly connected components

Top Cycle and Pareto-Optimality

- The top cycle is very large.
- In fact, it is so large that it fails to be Pareto-optimal when there are more than three alternatives (Ferejohn & Grether, 1977).

1	**1**	**1**
a	b	d
b	c	a
c	d	b
d	a	c

- Since Pareto-optimality is an essential ingredient of all Arrovian impossibilities, this escape route is (so far) not entirely convincing.

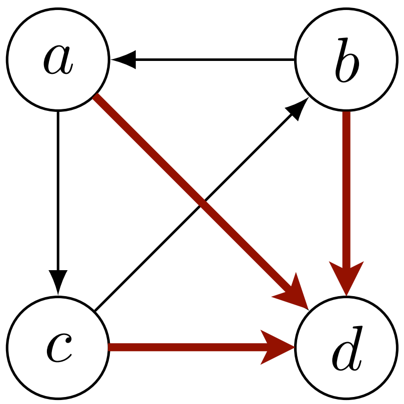
Peter C. Fishburn

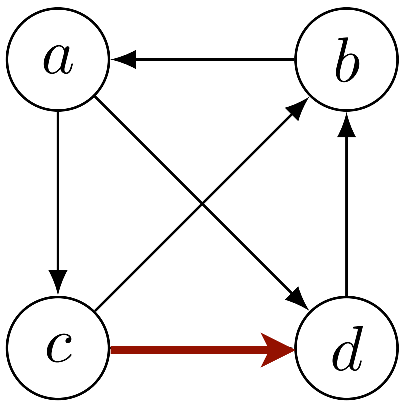
The Uncovered Set

Nicholas Miller

- Expansion can be further weakened to weak expansion: $S(A) \cap S(B) \subseteq S(A \cup B)$.
- Theorem (Moulin, 1986): There is a unique finest majoritarian SCF satisfying weak expansion: the uncovered set.
- Given a tournament (A,R_M), x covers y ($x\ C\ y$), if $D(y) \subset D(x)$.
  - Proposed independently by Fishburn (1977) and Miller (1980)
  - Transitive subrelation of majority rule
  - The uncovered set (UC) consists of all uncovered alternatives, i.e., $UC(A,P_M) = Max(C,A)$.
  - C rationalizes the uncovered set

Examples





$UC(A,R_M)=\{a,b,c\}$

$UC(A,R_M)=\{a,b,c\}$

$TC(A,R_M)=\{a,b,c,d\}$

Properties of the Uncovered Set

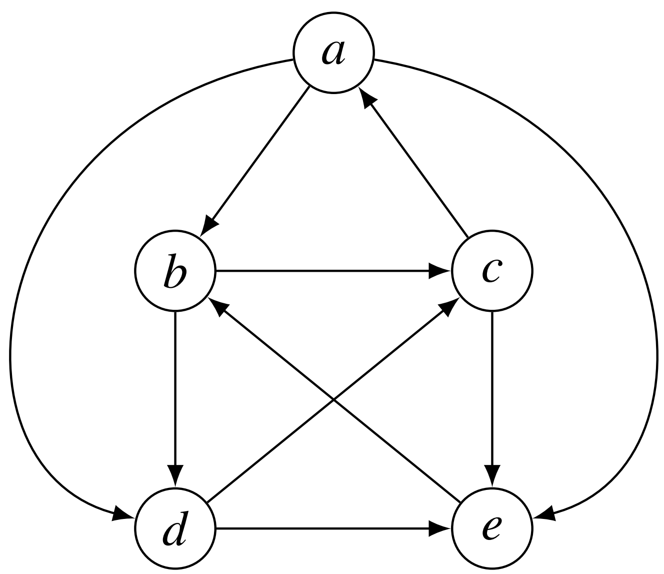
- Since expansion ⇒ weak expansion, $UC \subseteq TC$.
  - *UC* is a Condorcet extension.
- *UC* satisfies Pareto-optimality.
  - Theorem (B. and Geist, 2014): *UC* is the largest majoritarian SCF satisfying Pareto-optimality.
- How can the uncovered set be efficiently computed?
  - Straightforward $O(m^3)$ algorithm that computes the covering relation for every pair of alternatives
  - Can we do better than that?

Uncovered Set Algorithm

- Equivalent characterization of *UC*
  - Theorem (Shepsle & Weingast, 1984): *UC* consists precisely of all alternatives that reach every other alternative in at most two steps.
    - Such alternatives are called kings in graph theory.
- Algorithm via matrix multiplication
    - Fastest known matrix multiplication algorithm (Vassilevska Williams, 2011): $O(m^{2.3727})$
    - Strongly based on a previous algorithm (Coppersmith & Winograd, 1990): $O(m^{2.376})$
    - Matrix multiplication is believed to be feasible in linear time ($O(m^2)$).

procedure $UC(A, P_M)$
  for all $i, j \in A$ **do**
    if $i\ P_M\ j$ **then** $m_{ij} \leftarrow 1$
    else $m_{ij} \leftarrow 0$ **end if**
  end for
  $M \leftarrow (m_{ij})_{i,j \in A}$
  $U \leftarrow (u_{ij})_{i,j \in A} \leftarrow M^2 + M + I$
  $B \leftarrow \{i \in A \mid \forall j \in A : u_{ij} \neq 0\}$
  return B

Uncovered Set Algorithm (Example)



procedure $UC(A, P_M)$
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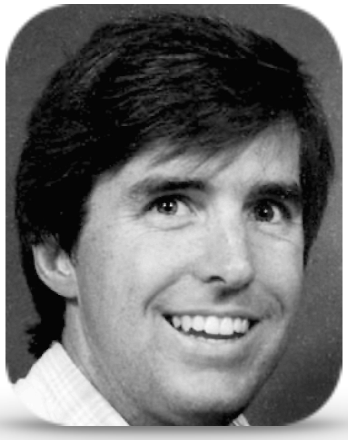
  return B

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Banks Set

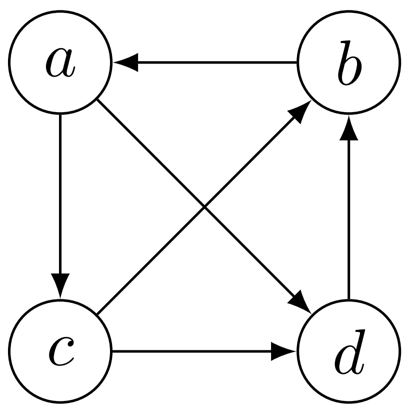


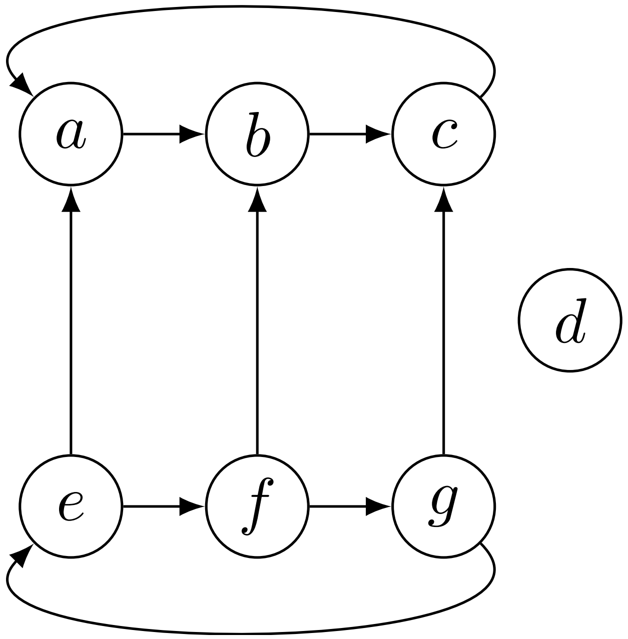
Jeffrey S. Banks

- Weak expansion can be weakened to strong retentiveness: $S(\overline{D}(x)) \subseteq S(A)$ for all $x \in A$.
- Theorem (B., 2011): There is a unique finest majoritarian SCF satisfying strong retentiveness: the Banks set.
- A *transitive subset* of a tournament (A, R_M) is a set of alternatives $B \subseteq A$ such that R_M is transitive within B.
- Let $Trans(A,R_M) = \{B \subseteq A \mid B \text{ is transitive}\}$.
- The Banks set (BA) consists of the maximal elements of all inclusion-maximal transitive subsets (Banks, 1985), i.e., $BA(A,R_M) = \{Max(R_M,B) \mid B \in Max(\supseteq, Trans(A,R_M))\}$

Examples

(All missing edges are pointing downwards.)





$UC(A,R_M)=\{a,b,c\}$

$BA(A,R_M)=\{a,b,c\}$

$TC(A,R_M)=\{a,b,c,d,e,f,g\}$

$UC(A,R_M)=\{a,b,c,d\}$

$BA(A,R_M)=\{a,b,c\}$

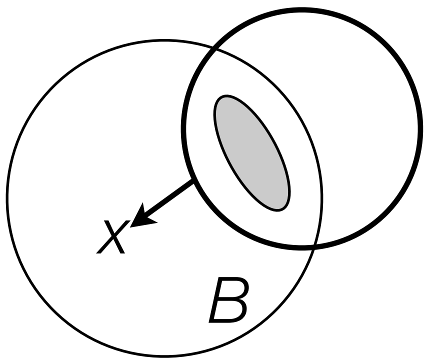
Properties of the Banks Set

- Since expansion ⇒ weak expansion ⇒ strong retentiveness, $BA \subseteq UC \subseteq TC$.
  - As a consequence, BA is a Condorcet extension and satisfies Pareto-optimality.
- Random alternatives in BA can be found in linear time by iteratively constructing maximal transitive sets.
- Yet, computing the Banks set is NP-hard (Woeginger, 2003) and remains NP-hard even for 7 voters (B. et al., 2013).
- Strong retentiveness can be further weakened to retentiveness:
  $S(\overline{D}(x)) \subseteq S(A)$ for all $x \in S(A)$.

Tournament Equilibrium Set

Thomas Schwartz

- Let S be an arbitrary choice function.
  - A non-empty set of alternatives B is *S-retentive* if $S(\overline{D}(x))\subseteq B$ for all $x\in B$.
  - Idea: No alternative in the set should be "properly" dominated by an outside alternative.



- $\mathring{S}$ is a new choice function that yields the union of all inclusion-minimal S-retentive sets.
  - $\mathring{S}$ satisfies retentiveness.
- The tournament equilibrium set (TEQ) of a tournament is defined as $TEQ=\mathring{TEQ}$.
  - recursive definition (unique fixed point of ring-operator)
  - Theorem (Schwartz, 1990): $TEQ\subseteq BA$.

Properties of TEQ

- Computing *TEQ* is NP-hard (B. et al., 2010) and remains NP-hard even for 9 voters (B. et al., 2013).
  - The best known upper bound is PSPACE!
- Theorem (Laffond et al., 1993; Houy 2009; B., 2011; B. and Harrenstein, 2011): The following statements are equivalent:
  - Every tournament contains a unique minimal *TEQ*-retentive set. (Schwartz' Conjecture, 1990)
  - *TEQ* is the unique finest majoritarian SCF satisfying retentiveness.
  - *TEQ* satisfies monotonicity (and many other desirable properties).
- All or nothing:
  Either *TEQ* is a most appealing SCF or it is severely flawed.

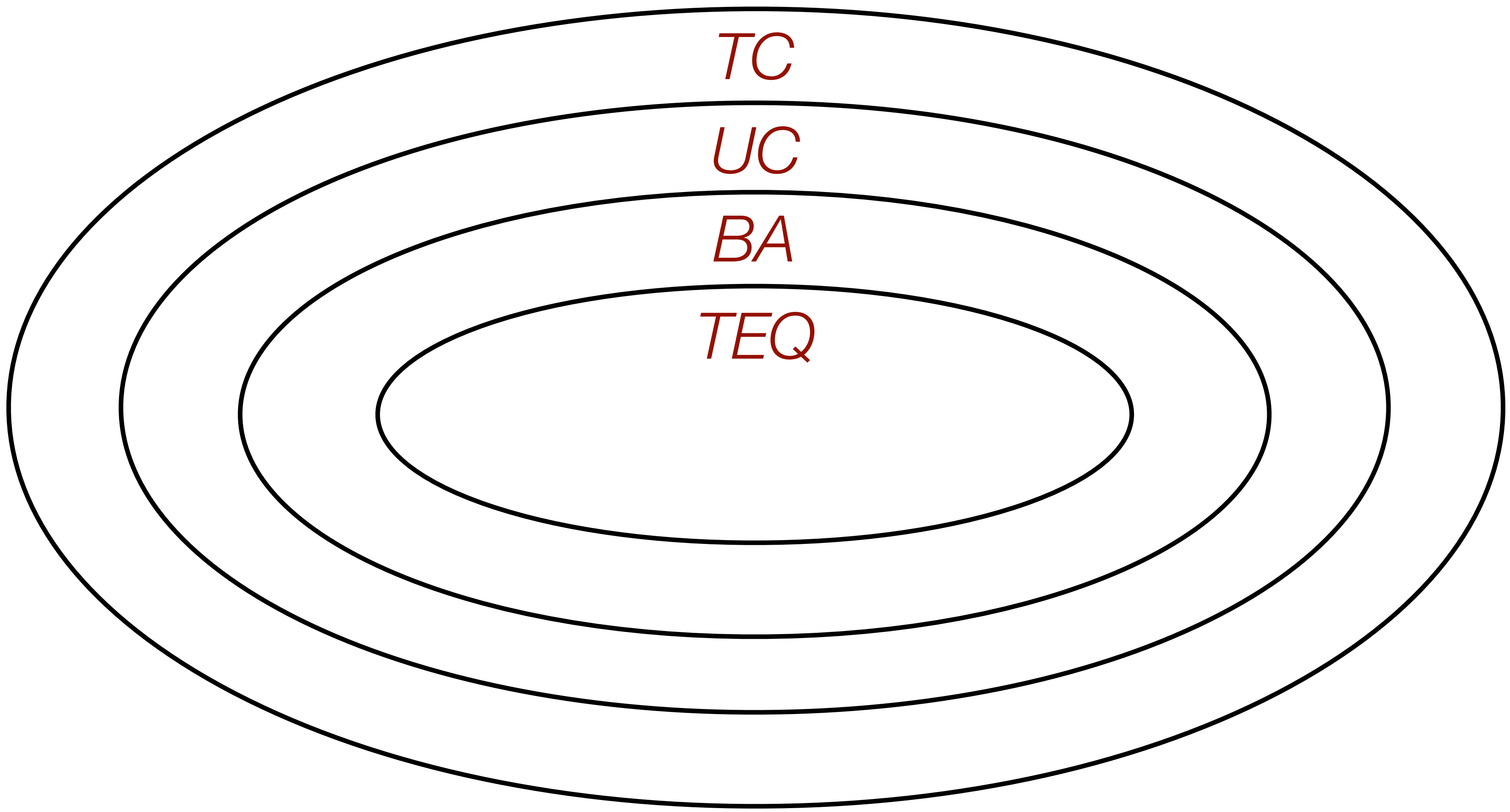
Schwartz's Conjecture

- There exists no counterexample with less than 13 alternatives (154 billion tournaments have been checked).
  - TEQ satisfies all nice properties if $|A|<13$.
- No counterexample was found by searching billions of random tournaments with up to 50 alternatives.
  - Checking significantly larger tournaments is intractable.
- Many non-trivial weakenings of Schwartz's conjecture are known to hold (Good, 1971; Dutta, 1988; B. et al., 2010; B., 2011).
- Theorem (B., Chudnovsky, Kim, Liu, Norin, Scott, Seymour, and Thomassé, 2012): Schwartz's conjecture is false.

Aftermath

- Non-constructive proof relying on a probabilistic argument by Erdős and Moser (1964)
  - Neither the counter-example nor its size can be deduced from proof.
  - Smallest counter-example of this type requires about 10^{136} alternatives.
- More recently, a counter-example with 24 alternatives was found with the help of a computer (B. & Seedig, 2013).
- In principle, *TEQ* is severely flawed. However, counter-examples are so extremely rare that this has no practical consequences.
  - This casts doubt on the axiomatic method.

Weakly Consistent SCFs



Top Cycle (1971)	*TC*	expansion	*O*
Uncovered Set (1977)	*UC*	weak expansion	*O*
Banks Set (1985)	*BA*	strong retentiveness	*2*
Tournament Equilibrium Set (1990)	*TEQ*	(retentiveness)	*2*

Escape Route #3

Randomization

Social Decision Schemes

- A *social decision scheme* (SDS) maps a preference profile to a lottery (probability distribution) $p \in \Delta(A)$ over the alternatives.
- Let $g(x,y) = n_{xy} - n_{yx}$ be the *majority margin* of x and y.
- Alternative x is a Condorcet winner if $g(x,y) \geq 0$ for all $y \in A$.
- g can be straightforwardly extended to an *expected majority margin* $g(p,q) = \sum_{x,y \in A} p(x) \cdot q(y) \cdot g(x,y)$.
- Lottery p is maximal if $g(p,q) \geq 0$ for all $q \in \Delta(A)$.
    - Maximal lotteries are guaranteed to exist due to the minimax theorem and are unique when $|N|$ is odd (Laffond et al., 1997).

Germaine Kreweras

Maximal Lotteries

Peter C. Fishburn

- First studied by Kreweras (1965) and Fishburn (1984)
  - rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)
- *g* can be seen as a symmetric two-player zero-sum game.
  - Maximal lotteries are mixed minimax strategies.
- Example

2	**2**	**1**
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

$$\begin{array}{c|ccc|} & a & b & c \\ \hline a & \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ b & \mathbf{-1} & \mathbf{0} & \mathbf{3} \\ c & \mathbf{1} & \mathbf{-3} & \mathbf{0} \\ \hline \end{array}$$

  - The unique maximal lottery is *3/5 a + 1/5 b + 1/5 c*.

Properties of Maximal Lotteries (*ML*)

- *ML* can be efficiently computed via LP.
- Pareto-dominated alternatives always get zero probability in every maximal lottery.
  - In fact, *ML* is even efficient with respect to stochastic dominance.
    - No lottery gives more expected utility for any utility representation consistent with the voters' preferences (Aziz et al., 2012).
- *ML* is weakly strategyproof in a well-defined sense (Aziz et al., 2012)
- *ML* can be uniquely characterized using appropriate generalizations of consistency and reinforcement (Brandl et al., forthcoming).
  - The "dilemma of social choice" is resolved via randomization!

Tutorial Summary

- Rational choice theory
- May's Theorem, Condorcet's Paradox, Arrow's Theorem
- Three escape routes:
  - replace consistency with a variable-electorate condition
    - Young's characterization of scoring rules (e.g., plurality, Borda)
  - weaken consistency
    - top cycle (expansion)
    - uncovered set (weak expansion)
    - Banks set (strong retentiveness)
    - tournament equilibrium set (retentiveness)
  - randomization
    - maximal lotteries (randomized Condorcet winners)

Recommended Literature

- Introductory book chapter
  - F. Brandt, V. Conitzer, and U. Endriss. *Computational Social Choice*. In "*Multiagent Systems*" (G. Weiss, ed.), MIT Press, 2013.
- Books
  - M. Allingham: *Choice Theory - A very short introduction*. Oxford University Press, 2002
  - D. Austen-Smith and J. Banks: *Positive Political Theory I*, University of Michigan Press, 1999
  - W. Gärtner: *A Primer in Social Choice Theory*, Oxford University Press, 2009
  - J.-F. Laslier: *Tournament Solutions and Majority Voting*. Springer-Verlag, 1997
  - H. Moulin: *Axioms of Cooperative Decision Making*. Cambridge University Press, 1988
  - S. Nitzan: *Collective Choice and Preference*, 2010